Cooperative filtering for parameter identification of diffusion processes

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Abstract—This paper presents a cooperative filtering scheme for online parameter identification of 2D diffusion processes using data collected by a mobile sensor network moving in the diffusion field. The diffusion equation is incorporated into the information dynamics associated with the trajectories of the mobile sensors. A cooperative Kalman filter is developed to provide estimates of field values, the gradient, and the temporal variations of the field values along the trajectories. This leads to a co-design scheme for state estimation and parameter identification for diffusion processes that is different from using static sensors. Utilizing the state estimates from the filters, a recursive least square (RLS) algorithm is designed to estimate the unknown diffusion coefficient of the field. A set of sufficient conditions is derived for the convergence of the cooperative Kalman filter. Simulation results show satisfactory performance of the proposed method.

I. INTRODUCTION

Many environmental processes are characterized by both spatial and temporal correlations and often represented mathematically by partial differential equations (PDEs). One of the typical PDEs is the diffusion equation, which has been widely used to model the diffusion phenomena such as the propagation of chemical contaminates in the water or in the air [1]. In many practical problems, some parameters in the diffusion equation such as the diffusion coefficient may be unknown or inaccurate, which requires identification or calibration [2]. For instance, in the application of smog modeling and prediction, the unknown diffusivity tensor in an advection-diffusion PDE can be estimated based on measurements collected by monitoring stations [3].

There exist many contributions on the issue of parameter identification of PDEs using static sensor networks [4]–[6]. However, in missions of modeling a relatively large region, the use of static sensor networks is often impractical due to the sheer size of the fields and high cost of installing enough static sensors to ensure coverage of the entire field [7] [8]. For parameter identification purpose, a preferable opportunity is using mobile sensor networks, which are collections of robotic agents with computational, communication, sensing, and locomotive capabilities [2]. There has been an immense interest in the use of mobile sensor networks to detect, monitor, and model the environment [2], [9]–[14]. One of the general approaches of parameter identification is to first decide optimal locations or trajectories of sensors offline, then, formulate a least square problem and search for the parameters that minimize the error between measurements of the true state (with true parameters) and the estimated state [2], [15]. This is usually referred to as performing the twin experiments in data assimilation literature [16]. To find parameters that minimize the least square cost function, PDEs have to be solved using finite element methods over the entire spatial domain, and the optimal solution is obtained by numerical methods for each time step, which requires high computational power. In many realistic scenarios, it is desirable and more practical to achieve parameter estimation while a mobile sensor network is exploring a field instead of estimating parameters offline. For example, in chemical plume tracking, the mobile sensor network has no prior knowledge of the diffusion process, thus, it’s preferable that the mobile sensor network can estimate the unknown diffusion coefficient while detecting and tracking the plume to obtain real-time information about the process. Therefore, we aim to develop an online parameter identification algorithm that estimates parameters iteratively, which can save tremendous computational power.

In this paper, we develop a cooperative filtering scheme for online parameter identification of diffusion processes using a mobile sensor network. We first incorporate the diffusion equation into the information dynamics and develop a cooperative Kalman filter. Compared to the cooperative Kalman filter in [10], the proposed filter deals with a diffusion field instead of a static field and involves more states, which enables the online estimation of the temporal variations of the field values along the trajectory of a mobile sensor network. Meanwhile, we provide a method to estimate the Hessian that is necessary for the cooperative Kalman filter. Utilizing the estimates from the filters, we employ the recursive least square (RLS) method to iteratively update the estimate of the unknown parameter in the diffusion equation. We justify a set of sufficient conditions regarding the formation shape and motion of the mobile sensor network that guarantee the convergence of the proposed filter. Finally, we present simulation results to show the satisfactory performance.

The problem is formulated in Section II. Section III presents the cooperative filtering design and Section IV shows the convergence proof. Simulation results are presented in Section V and conclusions follow in Section VI.

II. PROBLEM FORMULATION

A. Diffusion processes

Consider the two-dimensional (2D) diffusion process defined on a domain \( \Omega \subseteq \mathbb{R}^2 \):

\[
\frac{\partial z(r,t)}{\partial t} = \theta \nabla^2 z(r,t), \quad r \in \Omega,
\]

\( (1) \)
where \( z(r,t) \) is the concentration function, \( \theta > 0 \) is a constant diffusion coefficient, and \( \nabla^2 \) represents the Laplacian operator. In practical applications such as environmental monitoring, the exploration domain \( \Omega \) is much larger than the source and sensor dimensions so that the boundary can be modeled as a flat surface. Thus, we assume Dirichlet boundary conditions on the boundary \( \partial \Omega \) [17],

\[
z(r,t) = 0, \quad r \in \partial \Omega. \tag{2}
\]

Many natural processes can be described by the diffusion equation (1). In many scenarios, \( \theta \) is unknown or inaccurate, which requires identification or estimation.

**B. Sensor dynamics**

Consider a formation of \( N \) coordinated sensing agents moving in the field, each of which carries a sensor that takes point measurements of the field \( z(r,t) \). We consider the sensing agents with single-integrator dynamics given by

\[
r_i(t) = u_i(t), \quad i = 1, 2, \ldots, N, \tag{3}
\]

where \( r_i(t) \) and \( u_i(t) \in \mathbb{R}^2 \) are the position and the velocity of the \( i \)th agent, respectively.

In most applications, the sensor measurements are taken discretely over time. Let the moment when new measurements are available be \( t_k \), where \( k \) is an integer index. Denote the position of the \( i \)th agent at time \( t_k \) be \( r_i^k \) and the field value at \( r_i^k \) is \( z(r_i^k,k) \). Let \( r_c^k = \left[ r_{c,x}^k, r_{c,y}^k \right] \) be the center of the formation at time \( t_k \), i.e., \( r_c^k = \frac{1}{N} \sum_{i=1}^{N} r_i^k \). The measurement of the \( i \)th agent can be modeled as

\[
p(r_i^k,k) = z(r_i^k,k) + n_i, \tag{4}
\]

where \( n_i \) is assumed to be i.i.d. Gaussian noise. We have the following assumption for the sensing agents.

**Assumption II.1** Each agent can obtain its position \( r_i^k \) and the measurement of concentration value \( z(r_i^k,k) \), and share these information with other agents.

The problem is formulated as: under Assumption II.1, develop an online parameter identification algorithm that estimates the unknown constant diffusion coefficient \( \theta \) of the diffusion equation (1) based on the information collected by a mobile sensor network moving in the diffusion field.

**III. COOPERATIVE FILTERING FOR PARAMETER IDENTIFICATION**

This section describes our approach for parameter identification of the diffusion model (1). Before discussing the algorithm, we first introduce the numerical scheme for the diffusion equation. We then derive the information dynamics and construct a cooperative Kalman filter. Next, we show how to estimate the Hessian that is necessary for the cooperative Kalman filter. Finally, we show how to use the RLS method to iteratively update the estimate of \( \theta \).

**A. Numerical scheme for the diffusion equation**

In order to run RLS to estimate \( \theta \) based on discrete measurements, we discretize the diffusion PDE (1) at the formation center \( r_c^k \). The discretization of the PDE requires approximating spatial and temporal derivatives by means of finite difference method.

Assume the current time step is \( k \). Then the term \( \frac{\partial z(r_c^k,t)}{\partial t} \) can be approximated by

\[
\frac{\partial z(r_c^k,t)}{\partial t} \bigg|_{t=t_k} \approx \frac{z(r_c^k, k+1) - z(r_c^k, k)}{t_s}, \tag{5}
\]

and \( \nabla^2 z(r_c^k,t) \) can be expressed with a finite difference approximation as:

\[
\nabla^2 z(r_c^k,t) \approx \frac{\partial^2 z(r_c^k,t)}{\partial^2 r_{c,x}} + \frac{\partial^2 z(r_c^k,t)}{\partial^2 r_{c,y}} \\
\approx \frac{z(r_c^k - \Delta r_x \cdot x,k) - 2z(r_c^k,k) + z(r_c^k + \Delta r_x \cdot x,k)}{\Delta r_x^2} \\
+ \frac{z(r_c^k - \Delta r_y \cdot y,k) - 2z(r_c^k,k) + z(r_c^k + \Delta r_y \cdot y,k)}{\Delta r_y^2}, \tag{6}
\]

where \( t_s \) is the sampling time, \( x = [1 0]^T \) and \( y = [0 1]^T \) are unit axis vectors, \( \Delta r_x \) and \( \Delta r_y \) are the grid sizes along the \( x \) and \( y \) directions, \( z(r_c^k,k) \) is the field value at position \( r_c^k \) at the current time step \( k \), and \( z(r_c^k,k-1) \) is the field value at position \( r_c^k \) at the previous time step \( k-1 \). By employing formation control, we can arrange four agents in a symmetric formation as shown in Fig. 1. Then we have \( \Delta r_x = \| r_{c}^k - r_{c}^{k-1} \|, \Delta r_y = \| r_{c}^k - r_{c}^{k-1} \| = \| r_{c}^k - r_{c}^{k-1} \| \). For a symmetric formation, \( \Delta r_x = \Delta r_y \), which can be considered as a constant value. Substituting the values of \( \Delta r_x \) and \( \Delta r_y \) into Equation (6), we get

\[
\nabla^2 z(r_c^k,t) \approx \frac{z(r_{c,1}^k,k) + z(r_{c,2}^k,k) + z(r_{c,3}^k,k) + z(r_{c,4}^k,k) - 4z(r_c^k,k)}{\Delta r_r^2}, \tag{7}
\]

where \( z(r_{c,i}^k,k) \) for \( i = 1, 2, 3, 4 \) are field values at positions \( r_{c,i}^k \) at time step \( k \). Applying the finite differences (6), (7) to

![Fig. 1. Formation of four agents](image)

Equation (1) gives:

\[
\frac{z(r_c^k, k+1) - z(r_c^k, k)}{t_s} = \theta \cdot \sum_{i=1}^{4} z(r_{c,i}^k,k) - 4z(r_c^k,k) \frac{\Delta r_r^2}{\Delta r_c^2}. \tag{8}
\]

If we can get the estimates of \( z(r_c^k, k+1), z(r_c^k, k) \), and \( \sum_{i=1}^{4} z(r_{c,i}^k,k) \), then \( \theta \) can be estimated using RLS based on the discretized model (8). It should be noted that the sampling time \( t_s \) must obey the inequality \( t_s \leq \frac{\Delta r_r^2 + \Delta r_c^2}{4\theta} \) for the discretization method to converge [17].
Remark III.1 Even though we only consider four agents in the above derivative, it should be noted that our scheme can be readily extended to $N > 4$ case by following the implementation of standard finite-volume method.

B. The design of the cooperative Kalman filter

We first introduce the motivation of designing a cooperative filter by pointing out the difference between $z(r_c^k, k + 1)$ and $z(r_c^k, k)$ in (5). By designing a cooperative filter similar to the one developed in [10], $z(r_c^k, k)$ may be directly estimated by combining the measurements taken by the sensing agents at time step $k$. However, at time step $k + 1$, the formation center of the group is at position $r_c^{k+1}$. Therefore, the same filter can only provide the estimate of $z(r_c^{k+1}, k + 1)$, not $z(r_c^k, k + 1)$. In order to estimate the temporal variations of the field value in (5) along the trajectory of the mobile sensor network, we first need to derive a new cooperative filter to estimate both $z(r_c^k, k)$ and $z(r_c^k, k + 1)$.

To construct a cooperative Kalman filter to obtain the estimates of $z(r_c^k, k)$ and $z(r_c^k, k + 1)$, we first analyze the dynamics of the diffusion field value along the trajectory of the formation center $r_c$ according to

$$\dot{z}(r_c, t) = \frac{\partial z(r_c, t)}{\partial r_c} \frac{dr_c}{dt} + \frac{\partial z(r_c, t)}{\partial t} = \nabla z(r_c, t) \cdot \dot{r}_c + \frac{\partial z(r_c, t)}{\partial t},$$

(9)

where $\nabla z(r_c, t)$ is the gradient of $z(r_c, t)$. Substituting Equation (1) into (9), we obtain

$$\dot{z}(r_c, t) = \nabla z(r_c, t) \cdot \dot{r}_c + \theta \nabla^2 z(r_c, t).$$

(10)

Applying the finite differences to each term of (10) at time $t = t_{k-1}$ and at position $r_c = r_c^{k-1}$ gives:

$$\dot{z}(r_c, t) |_{t = t_{k-1}, r_c = r_c^{k-1}} = \frac{z(r_c^k, k) - z(r_c^{k-1}, k - 1)}{t_k},$$

$$\nabla z(r_c, t) |_{t = t_{k-1}, r_c = r_c^{k-1}} = \frac{(r_c^k - r_c^{k-1})^T \nabla z(r_c^{k-1}, k - 1)}{t_k},$$

$$\theta \nabla^2 z(r_c, t) |_{t = t_{k-1}, r_c = r_c^{k-1}} = \theta \sum_{i=1}^{4} \frac{z(r_c^{k-1}, k - 1) - 4z(r_c^{k-1}, k - 1) + (r_c^{k-1})^T \nabla z(r_c^{k-1}, k - 1)}{\Delta r_c^2}.$$

(11)

Substituting the finite difference equations (11) into Equation (10) gives the information dynamics of $z(r_c^k, k)$ as

$$z(r_c^k, k) = \left(1 - 4t_k \frac{\hat{\theta} k}{\Delta x^2}\right) z(r_c^{k-1}, k - 1) + t_k \hat{\theta} k \sum_{i=1}^{4} z(r_c^{k-1}, k - 1) + \frac{(r_c^k - r_c^{k-1})^T \nabla z(r_c^{k-1}, k - 1)}{t_k},$$

(12)

Similarly, we also obtain the dynamics of $z(r_c^k, k + 1)$ by discretizing Equation (10) at time $t = t_k$ and at position $r_c = r_c^{k}$. Then, denoting the measurement vector as

$$P(k) = [p(r_c^{k-1}, k - 1) \cdots p(r_c^{k-N}, k - 1) p(r_c^k, k) \cdots p(r_c^{k+M}, k)]^T,$$

which is modeled as,

$$P(k) = C(k) \cdot X(k) + D(k) \hat{H}(k) + v(k),$$

(20)

where $\hat{H}(k) = [\hat{H}(r_c^{k-1}, k - 1) \hat{H}(r_c^{k}, k)]^T$ represents the estimate of the Hessian at the center $r_c^{k}$ in a vector form,
\( v(k) \) is the zero-mean measurement noise with covariance matrix \( R \).

\[
C(k) = \begin{bmatrix}
1 & (r_{k} - r_{c}^{-1})^T & 0 & \cdots \\
1 & (r_{k} - r_{c}^{-1})^T & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 1 & (r_{N} - r_{c}^{-1})^T \\
\end{bmatrix},
\]

(21)
and

\[
D(k) = \begin{bmatrix}
\frac{1}{2}((r_{1} - r_{c}^{-1}) \otimes (r_{1} - r_{c}^{-1}))^T & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{2}((r_{N} - r_{c}^{-1}) \otimes (r_{N} - r_{c}^{-1}))^T & \cdots & 0 \\
\end{bmatrix},
\]

(22)

where \( \otimes \) is the Kronecker product. In this paper, we assume the covariance matrices \( Q \) and \( R \) are known through measurements or computation [10].

Moreover, from Equation (8), we observe that there is a constraint between the state \( z(r_{k}, k + 1) \) and \( z(r_{k}, k) \) at each step. Hence, we add a state equality constraint for the above Kalman filter:

\[
G(k) \cdot X(k) = d(k),
\]

(23)

where \( G(k) = [1 \ 0 \ (\frac{\partial \theta}{\partial \theta} - 1) \ 0] \) and \( d(k) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{4} z(r_{i}^{-1}, k - 1) \). We observe that the proposed Kalman filter is based on the time-varying information dynamics (16) with the state equality constraint (23). This type of Kalman filter with state equality constraint has been previously investigated in [18].

C. The cooperative Kalman filter equations

By following canonical procedures in [18], the equations for Kalman filter with state equality constraints are given as

1. The one-step prediction,

\[
\hat{X}^{-}(k) = A \hat{X}(k - 1) \hat{X}^{+}(k - 1) + U(k - 1); 
\]

(24)

where \( \hat{X}^{+}(k - 1) \) is the current constrained state estimate and \( \hat{X}^{-}(k) \) is a prior unconstrained state estimate.

2. Error covariance for the one-step prediction,

\[
P_{c}^{+}(k) = A \hat{P}(k - 1) \hat{P}^{+}(k - 1) A^T(k - 1) + Q; 
\]

(25)

3. Optimal gain,

\[
K(k) = P_{c}^{+}(k) C^T(k) [C(k) P_{c}^{-}(k) C^T(k) + R]^{-1}; 
\]

(26)

4. Updated unconstrained estimate,

\[
\hat{X}^{+}(k) = \hat{X}^{-}(k) + K(k) (P(k) - C(k) \hat{X}^{-}(k) - D(k) \hat{H}(k));
\]

(27)

5. Error covariance for the updated estimate

\[
P_{c}^{+}(k) = (I - K(k) C(k)) P_{c}^{-}(k); 
\]

(28)

6. Updated constrained estimate,

\[
\hat{X}^{+}(k) = \hat{X}^{+}(k) - P_{c}^{+}(k) G^T(k) [G(k) P_{c}^{-}(k) G^T(k)]^{-1} \\
\cdot (G(k) \hat{X}^{-}(k) - d(k)).
\]

(29)

D. Cooperative estimation of the Hessian

Estimates of \( z(r_{k}, k), z(r_{k}^{-1}, k - 1) \), and the Hessian \( \hat{H}(k) \) in Equation (18) are needed to enable the cooperative Kalman filter.

1. Estimates of \( z(r_{k}, k), z(r_{k}^{-1}, k - 1) \):

Since the sensor measurements \( p(r_{k}, k) \) and \( p(r_{k}^{-1}, k - 1) \) are available in the measurement vector \( P(k) \), one straightforward and simple way is to replace \( z(r_{k}, k) \) and \( z(r_{k}^{-1}, k - 1) \) with the sensor measurements \( p(r_{k}, k) \) and \( p(r_{k}^{-1}, k - 1) \).

2. Cooperative estimation of the Hessian:

Using the cooperative Kalman filter, we can obtain a prediction for \( X(k) \) as \( \hat{X}^{-}(k) = \hat{A} \hat{X}(k - 1) \hat{X}^{+}(k - 1) + U(k - 1) \). If we assume the number of sensor \( N \geq 4 \) and the formation is not colinear, we have \( P(k) = C(k) \cdot \hat{X}^{-}(k) + D(k) \hat{H}(k) \). The Hessian estimate can be solved by using the least mean square method, \( \hat{H}(k) = (D(k)^T D(k))^{-1} D(k)^T (P(k) - C(k) \hat{X}^{-}(k)) \).

E. Recursive Least Square Estimation

In this section, we use the RLS method to iteratively update the estimate of \( \theta \) in the diffusion equation (1). We do this using the information state \( X(k + 1) = [z(r_{k}^{+}, k), \nabla z(r_{k}^{+}, k), z(r_{k}^{-}, k + 1), \nabla z(r_{k}^{-}, k + 1)]^T \) obtained from the filter to calculate the temporal variations of the field value along the trajectory \( z(r_{k+1}^{+}, k+1) - z(r_{k}^{+}, k) \) at each step. On the other hand, by replacing \( \sum_{i=1}^{4} z(r_{i}^{1}, k) \) with the sensor measurements \( \sum_{i=1}^{4} p(r_{i}, k) \), we can estimate the discrete Laplacian operator directly using sensor measurements \( P(k) \) from the agents according to Equation (7). Therefore, the diffusion coefficient can be directly estimated without the need of numerically solving the diffusion equation. Given an initial estimate for the diffusion coefficient, a simple application of the RLS method can iteratively update the estimate of \( \theta \). Following the canonical procedure of RLS estimation outlined in [19], we derive the following equations

\[
\hat{\theta}_{k} = \hat{\theta}_{k - 1} + g(k) \left( \frac{z(r_{k}^{+}, k) - z(r_{k}^{-}, k)}{\tau_s} - \nabla^2 z(r_{k}^{+}, k) \hat{\theta}_{k - 1} \right), 
\]

\[
g(k) = \eta(k - 1) \nabla^2 z(r_{k}^{+}, k) \left[ \nabla^2 z(r_{k}^{+}, k) \eta(k - 1) \nabla^2 z(r_{k}^{+}, k)^T + R_{\alpha} \right]^{-1},
\]

\[
\eta(k) = (I - g(k) \nabla^2 z(r_{k}^{+}, k) \eta(k - 1)),
\]

where \( g(k) \) is the estimator gain matrix, \( \eta(k) \) is the estimation error covariance matrix, and \( R_{\alpha} \) is the noise covariance.

F. Formation and motion control

Control laws for the velocities of the agents are required so that the mobile sensor network can move along a certain trajectory while maintaining a desired formation. We view the entire formation as a deformable body. Thus, there are two parts of control: motion control and formation control. With the gradient estimates provided by the cooperative Kalman filter, the motion control for the agents can be easily realized by setting the velocities of the agents to be aligned with the estimated gradient direction. Thus, the mobile sensor network can achieve simultaneously parameter estimation and gradient climbing. Furthermore, there exists several results about the formation control for mobile agents

4330
We omit the detailed design of control here due to space limitation. Interested readers can refer to [10] [20].

IV. CONVERGENCE ANALYSIS OF THE COOPERATIVE KALMAN FILTER

In this section, we prove the convergence of the cooperative Kalman filter. Theorem 7.4 in [21] states that if the time-varying system dynamics are uniformly complete controllable and uniformly complete observable, the Kalman filter for this system converges. With this result, we will establish a set of sufficient conditions for the mobile sensors such that the uniformly complete controllability and observability of the Kalman filter can be guaranteed. Let \( \Phi(k,j) \) be the state transition matrix from time \( t_j \) to \( t_k \), where \( k > j \). Then, \( \Phi(k,j) = A_\theta(k-1)A_\theta(k-2) \cdots A_\theta(j) \) and \( \Phi(k,j) = \Phi^{-1}(j,k) \). We have the following lemma.

**Lemma IV.1** For \( \Phi(k,j) \) as defined above and \( C(k) \) as defined in (21), we can have

\[
\Phi(k,j) = \begin{bmatrix}
\xi_0 & (r_c^{k-1} - r_c^{j-1})^T & 0 & 0 \\
0 & I_{d+2} & 0 & 0 \\
0 & 0 & \xi_0 & (r_c^{k-1} - r_c^{j-1})^T \\
0 & 0 & 0 & I_{d+2}
\end{bmatrix},
\]

(30)

and

\[
C(k)\Phi(k,j) = \begin{bmatrix}
\xi_0 & (r_c^{k-1} - r_c^{j-1})^T & 0 & 0 \\
0 & I_{d+2} & 0 & 0 \\
0 & 0 & \xi_0 & (r_c^{k-1} - r_c^{j-1})^T \\
0 & 0 & 0 & I_{d+2}
\end{bmatrix},
\]

(31)

where \( \xi_0 = (1 - \frac{4}{3\tau_2} \hat{\theta}_k - 1)(1 - \frac{4}{3\tau_2} \hat{\theta}_{k-1}) \cdots (1 - \frac{4}{3\tau_2} \hat{\theta}_j) \).

Let’s first restate the definitions of uniformly complete controllability and uniformly complete observability, respectively (modified from Definitions in [21]).

**Definition IV.2** The proposed cooperative filter is uniformly complete controllable if there exist \( \tau_1 > 0, \lambda_1 > 0 \), and \( \lambda_2 > 0 \) such that the controllability Grammian \( C(k, k - \tau_1) = \sum_{j=k-\tau_1}^{\tau_1} \Phi(k,j)\Phi(k,j)^T \) satisfies \( \lambda_1 I_{d+6} \leq C(k, k - \tau_1) \leq \lambda_2 I_{d+6} \) for all \( k > \tau_1 \). \( Q \) is the covariance for the noise \( v(k) \).

In the following procedures, there exist some positive real numbers \( \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \). All of these real numbers are time-independent bounds for various quantities, the values of which do not affect the correctness of our discussions. Note that, in this paper, a relation between two symmetric matrices \( A_1 \leq A_2 \) means that for any vector \( s \) with compatible dimension, there exists \( s^T A_1 s \leq s^T A_2 s \). We have the following lemma for uniformly complete controllability.

**Lemma IV.3** The proposed filter is uniformly complete controllable if the following conditions are satisfied:

- \( (Cd1) \) The covariance matrix \( Q \) is bounded, i.e., \( \lambda_3 I \leq Q \leq \lambda_4 I \) for some constants \( \lambda_3, \lambda_4 > 0 \).

- \( (Cd2) \) The speed of each agent is uniformly bounded, i.e., \( ||r_i^j - r_i^{j-1}|| \leq \lambda_5 \) for all time \( j \), for \( i = 1, \cdots, N \), and for some constant \( \lambda_5 > 0 \).

(Cd3) The estimated parameter \( \hat{\theta}_j \) is bounded, i.e., \( 0 \leq \hat{\theta}_j \leq \lambda_6 \). By properly selecting the sampling interval \( \tau_4 \) and formation size \( \Delta r \), we can make \( \hat{\theta}_j \) satisfy that \( 0 < 1 - \frac{4\lambda_6}{3\tau_2} \hat{\theta}_j \leq 1 \) for all time \( j \), which means \( \lambda_6 < \frac{4\lambda_6^2}{3\tau_2} \).

**Proof:** Based on condition (Cd1), we obtain that the controllability Grammian satisfies \( \lambda_3 \sum_{j=k-\tau_1}^{\tau_1} \Phi(k,j)\Phi(k,j)^T \leq C(k, k - \tau_1) \leq \lambda_2 \sum_{j=k-\tau_1}^{\tau_1} \Phi(k,j)\Phi(k,j)^T \) for any \( k > \tau_1 \). Therefore, if we can find the uniform bounds for each of these semi-definite symmetric matrices, i.e., \( \Phi(k,j)\Phi(k,j)^T \), the overall bound for the controllability Grammian can be obtained readily. We apply Lemma IV.1 to compute \( \Phi(k,j)\Phi(k,j)^T \), i.e.,

\[
\Phi(k,j)\Phi(k,j)^T = \begin{bmatrix}
\xi_0 + ||\hat{\delta}(r,k)||^2 & (\hat{\delta}(r,k))^T \\
(\hat{\delta}(r,k)) & I_{d+2}
\end{bmatrix},
\]

(32)

where we denote \( \hat{\delta}(r,k) = r^{k-1} - r^{j-1} \).

Using basic linear algebra, we can obtain the minimum eigenvalue of matrix (32) as \( \lambda_{min} = \frac{1}{2} \left( ||\hat{\delta}(r,k)||^2 + \xi_0 + 1 - \sqrt{||\hat{\delta}(r,k)||^2 + \xi_0 + 1} \right) \), and the maximum eigenvalue as \( \lambda_{max} = \frac{1}{2} \left( ||\hat{\delta}(r,k)||^2 + \xi_0 + 1 + \sqrt{||\hat{\delta}(r,k)||^2 + \xi_0 + 1} \right) \). By definition, we know the \( \hat{\delta}(r,k) \) is the averaged movement over all agents between time \( j \) and \( k \). Then, the condition (Cd2) indicates that \( ||\hat{\delta}(r,k)|| \leq (k-j)\lambda_5 \leq \tau_1 \lambda_5 \) for all \( i \in [k - \tau_1, k] \). Due to condition (Cd3), we can see that \( 0 < \xi_0 \leq 1 \). Since the term \( (||\hat{\delta}(r,k)||^2 + \xi_0 + 1)^2 - 4 \xi_0 \geq (\xi_0 - 1)^2 \geq 0 \), it can be proven readily that \( \lambda_{min} \) reaches its minimum value when \( ||\hat{\delta}(r,k)|| = \tau_1 \lambda_5 \) and \( \xi_0 \) approaches zero. That means the minimum value of \( \lambda_{min} \) is \( \lambda_2 = \frac{1}{2} \left( ||\tau_1 \lambda_5||^2 + 1 - \sqrt{||\tau_1 \lambda_5||^2 + 1} \right) = 0 \). On the other hand, \( \lambda_{max} \) assumes its minimum value when \( ||\hat{\delta}(r,k)|| = \tau_1 \lambda_5 \) and \( \xi_0 \) is one. This maximum value is \( \lambda_8 = \frac{1}{2} \left( ||\tau_1 \lambda_5||^2 + 2 \sqrt{||\tau_1 \lambda_5||^2 + 2} \right) > 0 \). Therefore, we can conclude that \( \lambda_{min} I_{d+6} \leq \Phi(k,j)\Phi(k,j)^T \leq \lambda_{max} I_{d+6} \) for all time \( j \in [k - \tau_1, k] \). Hence, \( \lambda_3 \lambda_5 \tau_1 I_{d+6} \leq C(k, k - \tau_1) \leq \lambda_4 \lambda_5 \tau_1 I_{d+6} \). Let \( \lambda_1 = \lambda_3 \lambda_5 \tau_1 \) and \( \lambda_2 = \lambda_4 \lambda_5 \tau_1 \). Thus, according to Definition IV.2, we have proved the uniformly complete controllability claim.

**Definition IV.4** The proposed cooperative filter is uniformly complete observable if there exist \( \tau_2 > 0, \lambda_9 > 0, \) and \( \lambda_{10} > 0 \) such that the observability Grammian \( \mathcal{O}(k, k - \tau_2) = \sum_{j=k-\tau_2}^{\tau_2} \Phi^T(j,k)C(j)\Phi(j,k) \) satisfies \( \lambda_{10} I_{d+6} \leq \mathcal{O}(k, k - \tau_2) \leq \lambda_{10} I_{d+6} \) for all \( k > \tau_2 \). Here \( R \) is the covariance for the measure noise \( v(k) \).

To prove the uniformly complete observability, we require one elementary Lemma IV.5 [10] and the Lemma IV.6 to establish the sufficient conditions for a moving formation.

**Lemma IV.5** Suppose two \( 2 \times 1 \) vectors \( a = [a_1 \ a_2]^T \) and \( b = [b_1 \ b_2]^T \) form an angle \( \Psi \) such that \( 0 < \Psi < \pi \). Then the
minimum eigenvalue $\lambda_{\min}$ of the $2 \times 2$ matrix $M = a \cdot a^T + b \cdot b^T$ is strictly positive, i.e. $\lambda_{\min} > 0$

**Lemma IV.6** The proposed Kalman filter is uniformly completely observable if (Cd2), (Cd3) and the following conditions are satisfied:

(Cd4) The number of agents $N$ is greater than or equal 4.

(Cd5) The covariance matrix $R$ is bounded, i.e., $\lambda_{11} I \leq R \leq \lambda_{12} I$ for some constants $\lambda_{11}, \lambda_{12} > 0$.

(Cd6) The distance between each agent and the formation center is uniformly bounded from both above and below, i.e., $\lambda_{13} \leq \|r_i - r_j\| \leq \lambda_{14}$ and $\lambda_{15} \leq \|r_i - r_j\| \leq \lambda_{16}$ for all $j$, for $i = 1, 2, \ldots, N$, and for some constants $\lambda_{13}, \lambda_{14} > 0$.

(Cd7) There exists a constant time difference $\tau_2$, and for all $k > \tau_2$, there exists a time instance $j_k \in [k - \tau_2, k]$, as well as two agents indexed by $i_1$ and $i_2$, such that the following condition is satisfied: The two vectors, $r_{i_1} - r_{c_1}$ and $r_{i_2} - r_{c_2}$ form an angle $\psi_1$ that is uniformly bounded away from 0 or $\pi$.

(Cd8) There exist the time instance $j_2$ and two agents $i_1$ and $i_2$ as given by (Cd7), such that the following condition is satisfied: The two vectors, $r_{i_1} - r_{c_1}$ and $r_{i_2} - r_{c_2}$ form an angle $\psi_2$ that is uniformly bounded away from 0 or $\pi$.

**Proof:** Based on condition (Cd5), we first observe that the observability Gramian satisfies $\lambda_{17} \sum_{k=1}^{N} \Phi(j, k) C(j) C(j) \Phi(j, k) \leq \Sigma(k, k - \tau_2)$ and $\Sigma(k, k - \tau_2) \leq \lambda_{18} \sum_{k=1}^{N} \Phi(j, k) C(j) C(j) \Phi(j, k)$ for any $k$ and $\tau_2$ such that $k > \tau_2$. Then the uniformly complete observability can be proved by finding the positive upper and lower bounds for $\sum_{k=1}^{N} \Phi(j, k) C(j) C(j) \Phi(j, k)$ for all $k > \tau_2$. According to Lemma IV.1, we can get

$$\Phi(j, k) C(j) (\Phi(j, k)) = \begin{bmatrix} N \xi_{\theta}^2 & N \xi_{\theta} (r_{i_1} - r_{c_1})^T \\ N \xi_{\theta} (r_{i_1} - r_{c_1}) & N_i \end{bmatrix}.$$

Due to conditions (Cd2) and (Cd6), we can observe that each element of the above matrix is bounded above, i.e., $\Phi(j, k) C(j) C(j) \Phi(j, k) \leq \lambda_{17}$ for some constant $\lambda_{17} > 0$. For the lower bound, we can use conditions (Cd4), (Cd7) and (Cd8) to prove that there exists the lower bound $\lambda_{18} > 0$ such that $\lambda_{18} \leq \sum_{k=1}^{N} \Phi(j, k) C(j) C(j) \Phi(j, k)$. Since $\Phi(j, k) = \Phi(j, k) \Phi(j, k)^T C(j) C(j) \Phi(j, k)$, we have

$$l = \Phi(j, k) \Phi(j, k)^T C(j) C(j) \Phi(j, k) \Phi(j, k) = \Phi(j, k) \Phi(j, k)^T C(j) C(j) \Phi(j, k) = \Phi(j, k) I_\Phi(j, k),$$

where $\Phi(j, k) C(j) C(j) \Phi(j, k) = \Phi(j, k)$. By using the fact that $\sum_{i=1}^{N} (r_{i} - r_{c})^T = 0$, we have

$$C(j) C(j) = \begin{bmatrix} N & 0 \\ 0 & 0 \\ 0 & N \end{bmatrix}.$$

Therefore, using Lemma IV.5, we can readily get that $I_3$ is a positive semi-definite matrix. The Weyl’s theorem [10] can then be applied to prove that there exists
\( \lambda_{20} > 0 \) such that \( I_1 \geq \lambda_{20}I_{6 \times 6} \). Hence, \( O(k,k - \tau_2) \) is uniformly bounded below by \( O(k,k - \tau_2) \geq \lambda_9I_{6 \times 6} \).

V. SIMULATION RESULTS

To demonstrate the performance of the proposed approach for online parameter estimation for diffusion processes, we consider a well-known 2D diffusion equation (1) with the nominal value of \( \theta = 0.6 \). The initial condition is illustrated in Fig. 2(a), in which the maximum value is at point (20,30). The whole domain of PDE is a rectangular area \( 0 \leq x \leq 70, 0 \leq y \leq 90 \). We implement an implicit ADI finite-difference scheme in MATLAB, with 100-by-100 spatial grid. We add 1\% (in variance) white noise to measurements taken by sensors. A computational time step of 0.1s is chosen for the simulation, which satisfies the stability requirement of finite-difference method. In the simulation, we select the initial locations of four sensing agents represented by the red, blue, green, and purple stars as shown in Fig. 2(a). At each time step, the sensing agents take measurements of the field, run the proposed cooperative Kalman filter as well as the RLS algorithm to obtain the estimates of the diffusion coefficient \( \theta \), and move along the gradient direction estimated by the cooperative Kalman filter while converging to a desired formation. The trajectory of the center of the formation is shown in Fig. 2 (the blue dots). Initially, we set the estimate \( \hat{\theta}_0 = 2 \). The result of the proposed method is presented in Fig. 3. As we can observe from the figure, \( \hat{\theta} \) converges to the nominal value.

VI. CONCLUSION

We propose a novel filtering scheme for performing online parameter estimation for diffusion processes utilizing a mobile sensor network. Utilizing the filtered state as inputs, we employ the RLS algorithm to realize online estimation of the diffusion coefficient. Theoretical justifications are provided for the convergence analysis of the cooperative filter. Simulation results show satisfactory performance. Future work includes extending the proposed algorithm to spatially variant PDE models and demonstrating the methods in experiments involving robotic mobile sensor platforms.

REFERENCES


